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2000 J. Phys. A: Math. Gen. 33 8409

(http://iopscience.iop.org/0305-4470/33/47/307)

J. Phys. A: Math. Gen. **33** (2000) 8409–8425. Printed in the UK PII: S0305-4470(00)14533-0

Wave solutions of evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians

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Received 8 June 2000, in final form 12 October 2000

Abstract. The algebraic–geometric approach is extended to study evolution equations associated with the energy-dependent Schrödinger operators having potentials with *poles* in the spectral parameter, in connection with Hamiltonian flows on nonlinear subvarieties of Jacobi varieties. The general approach is demonstrated by using new parametrizations for constructing quasi-periodic solutions of the shallow-water and Dym-type equations in terms of theta-functions. A qualitative description of real-valued solutions is provided.

1. Introduction

The quasi-periodic solutions of most classical integrable PDEs can be obtained using the inverse spectral transform method (see, for review, Dubrovin 1981, Ablowitz and Segur 1981, Novikov *et al* 1984, Newell 1985, Ablowitz and Clarkson 1991). This is done by establishing a connection with an isospectral eigenvalue problem for an associated Schrödinger operator.

The algebraic–geometric technique was also developed for studying solutions of nonlinear evolution equations. One of the applications of this approach can be summarized as follows. By using the trace formula, families of quasi-periodic and soliton solutions are associated with Hamiltonian flows on finite-dimensional phase spaces. These flows are described by using so-called μ-variable representations leading to an Abel–Jacobi mapping, which include holomorphic and, in some cases, meromorphic differentials (see, amongst others, Novikov 1974, Lax 1975, Its and Kotlyarov 1976, Alber 1979, Ercolani 1989, Alber and Alber 1985, Belokolos *et al* 1994). Then the mapping is inverted in terms of Riemann theta-functions and their singular limits. Many well known nonlinear equations such as KdV, sine–Gordon, focusing and defocusing nonlinear Schrödinger equations, which describe a wide variety of important phenomena in physics, optics, biology and engineering, have been studied by using this approach.

Recently special attention was given to the shallow-water (SW) equation derived by Camassa and Holm (1993) in the context of the Hamiltonian structure,

$$U_t + 3UU_x = U_{xxt} + 2U_xU_{xx} + UU_{xxx} - 2\kappa U_x \tag{1.1}$$

and the Dym-type equation (see Cewen 1990, Hunter and Zheng 1994, Alber *et al* 1994, 1995)

$$U_{xxt} + 2U_x U_{xx} + U U_{xxx} - 2\kappa U_x = 0 \qquad \kappa = \text{const.} \tag{1.2}$$

Camassa and Holm (1993) described classes of n-peakon soliton-type solutions for an integrable (SW) equation (1.1). In particular, they obtained a system of completely integrable Hamiltonian equations for the locations of the 'peaks' of the solution, the points at which its spatial derivative changes sign. In other words, each peakon solution can be associated with a mechanical system of moving particles. Calogero (1995) and Calogero and Francoise (1996) further extended the class of mechanical systems of this type. The r-matrix approach was applied to the Lax pair formulation of an n-peakon system by Ragnisco and Bruschi (1996), who also pointed out the connection of this system with the classical Toda lattice. A discrete version of the Adler–Kostant–Symes factorization method was used by Suris (1996) to study a discretization of the peakon lattice, realized as a discrete integrable system on a certain Poisson submanifold of $\mathrm{gl}(n)$ equipped with an r-matrix Poisson bracket. Beals *et al* (1999, 2000) used the Stieltjes theorem on continued fractions and the classical moment problem for studying multi-peakon solutions of the SW equation. Generalized peakon systems are described for any simple Lie algebra by Alber *et al* (2000b).

The problem of describing complex quasi-periodic solutions of the equations (1.1) and (1.2) can be reduced to solving finite-dimensional Hamiltonian systems on symmetric products of hyperelliptic curves. Namely, according to Alber *et al* (1994, 1995, 1999), such solutions can be represented in the following form:

$$U(x, t) = \mu_1 + \cdots + \mu_g - M \tag{1.3}$$

where g is a positive integer, M is a constant and the evolution of the variables μ is given by the equations

$$\sum_{i=1}^{g} \frac{\mu_i^k \, \mathrm{d}\mu_i}{2\sqrt{R(\mu_i)}} = \begin{cases} 0 & k = 1, \ldots, g-2 \\ \mathrm{d}t & k = g-1 \\ \mathrm{d}x & k = g. \end{cases} \tag{1.4}$$

Here $R(\mu)$ is a polynomial of degree $2g + 2$ (for the shallow-water equation (1.1)) or $2g + 1$ (for the Dym-type equation (1.2)). Also $M = 0$ for the Dym-type equation.

In contrast to the finite-dimensional reductions of such equations as KdV and sine–Gordon equations, system (1.4) contains a meromorphic differential. Also, the number of holomorphic differentials is less than the genus g of the corresponding hyperelliptic curve: $W^2 = R(\mu)$. This implies that the problem of inversion (1.4) cannot be solved in terms of meromorphic functions of x and t. Examples of such equations arise in several problems of mechanics. These were considered by Vanhaecke (1995) and Abenda and Fedorov (2000), where a connection was established with the flows on nonlinear subvarieties of hyperelliptic Jacobian varieties, so-called *strata*. In Alber *et al* (1997) a whole class of N-component systems with poles was shown to be integrable by reducing them to similar nonstandard inversion problems which contained meromorphic differentials. Therefore N-component systems can be overdetermined, implying that the genus of the spectral curve can be higher than the number of μ-variables.

Quasi-periodic solutions of the Dym equation were studied by Dmitrieva (1993a) and Novikov (1999) by using a connection with KdV equation and introducing additional phase functions. Soliton solutions of the Dym-type equation were studied by Dmitrieva (1993b). Periodic solutions of the shallow-water equation were discussed by McKean and Constantin (1999). The complex geometry of the travelling wave solutions, cusp and peakon solutions was previously studied by Alber *et al* (1994, 1995, 1999, 2000a) in connection with geodesic flows on Riemannian manifolds and by Li and Olver (1998) from the point of view of singularity analysis.

The main goal of this paper is to describe explicit formulae in terms of theta-functions and their singular limits for quasi-periodic and soliton-like solutions to the shallow-water equation (1.1) and Dym-type equation (1.2). We also explain the role of the 'mysterious' phase functions used by Dmitrieva (1993a) and Novikov (1999) when studying quasi-periodic solutions of the equation of the Dym type.

Usually in the case of integrable evolution equations quasi-periodic flows are linearized on the Jacobi varieties. In this paper we show that in the case of N-component systems with poles the x- and t-flows take place on *nonlinear* subvarieties (strata) of generalized (noncompact) Jacobians. This makes the above nonlinear equations quite different from such well known equations as KdV, sine–Gordon and nonlinear Schrödinger equations. For the sake of clarity, in this paper we describe solutions related to hyperelliptic curves of genus 2. For the formulae in the general n-dimensional case, see Alber and Fedorov (2000).

This paper is organized as follows. In section 2 we demonstrate the main difference between the nonlinear SW and Dym-type equations and the KdV equation from the point of view of the algebraic–geometric approach by obtaining explicit expressions in terms of theta-functions for the *stationary* quasi-periodic solutions in the genus 2 case. This is done by using new complex parametrizations.

In section 3 we find time-dependent solutions by integrating and inverting equations (1.4) in the genus 2 case. We show that these equations can be extended to a standard Abel–Jacobi mapping of a symmetric product of three copies of the hyperelliptic curve to its *generalized* Jacobian. The original system (1.4) then defines a mapping onto a two-dimensional nonlinear stratum of the Jacobian, a generalized theta-divisor, where the dynamics actually takes place. By fixing t in the expression for the solution in terms of theta-functions, we then recover the stationary quasi-periodic solutions obtained in section 2.

Section 4 contains qualitative analysis of real bounded solutions for the case when the Weierstrass points of the spectral curve are real.

Finally, notice that the general case of g-phase ($2 \leqslant g$) solutions of the SW and Dym-type equations as well as their different singular limiting cases are described in detail by Alber and Fedorov (2000) and Alber and Miller (2001).

2. Stationary quasi-periodic solutions

The main difference from the point of view of the algebraic–geometric approach between N-component systems associated with the energy-dependent Schrödinger operators having potentials with *poles* and more traditional completely integrable equations, such as the KdV equation, can be demonstrated by using genus 2 quasi-periodic solutions of the SW and Dym equations. One needs to consider them both because these two equations provide examples of two different subclasses of systems which should be treated differently.

We start by describing stationary solutions which describe profiles of the quasi-periodic wave solutions of the two equations. For the sake of clarity, in this paper we restrict ourselves to the simplest nontrivial case $g = 2$.

Stationary quasi-periodic solutions for the SW equation. According to the trace formula (1.3), in the genus 2 case we have

$$U(x, t) = \mu_1 + \mu_2 - \sum_{j=1}^{5} a_j \qquad (2.1)$$

and equations (1.4) take the form

$$\begin{aligned}
\frac{\mu_1\,\mathrm{d}\mu_1}{2\sqrt{R_6(\mu_1)}} + \frac{\mu_2\,\mathrm{d}\mu_2}{2\sqrt{R_6(\mu_2)}} &= \mathrm{d}t \\
\frac{\mu_1^2\,\mathrm{d}\mu_1}{2\sqrt{R_6(\mu_1)}} + \frac{\mu_2^2\,\mathrm{d}\mu_2}{2\sqrt{R_6(\mu_2)}} &= \mathrm{d}x
\end{aligned} \tag{2.2}$$

where

$$R_6(\mu) = -\kappa\mu(\mu - a_1)\dots(\mu - a_5) \qquad a_1, \dots, a_5 = \text{const}.$$

(For details see Alber *et al* 1994.) Here we suppose that all the roots of $R_6(\mu)$ are distinct. The variables μ_1 and μ_2 must be regarded as coordinates of points $P_1 = (\mu_1, w_1)$, $P_2 = (\mu_2, w_2)$ on the genus 2 hyperelliptic curve $\Gamma = \{w^2 = R_6(\mu)\}$. Equations (2.2) involve one holomorphic differential and one meromorphic differential of the third kind having a pair of simple poles at the infinite points ∞_- and ∞_+ on Γ. Integrating (2.2), we obtain the mapping of the symmetric product $\Gamma^{(2)}$ to $\mathbb{C}^2 = (t, x)$:

$$\begin{aligned}
\int_{P_0}^{P_1} \frac{\mu\,\mathrm{d}\mu}{2\sqrt{R_6(\mu)}} + \int_{P_0}^{P_2} \frac{\mu\,\mathrm{d}\mu}{2\sqrt{R_6(\mu)}} &= t \\
\int_{P_0}^{P_1} \frac{\mu^2\,\mathrm{d}\mu}{2\sqrt{R_6(\mu)}} + \int_{P_0}^{P_2} \frac{\mu^2\,\mathrm{d}\mu}{2\sqrt{R_6(\mu)}} &= x
\end{aligned} \tag{2.3}$$

where P_0 is a fixed basepoint of the mapping. Notice that choosing P_1 or $P_2 = \infty_-, \infty_+$, yields $x = \infty$. Let us fix a canonical basis of cycles A_1, A_2, B_1, B_2 on Γ in a standard way (see, for example, Mumford 1983). The mapping has associated with it four independent (over the reals) two-dimensional vectors of periods of the differentials described above, along these cycles. In addition, it has one extra period vector corresponding to a zero-homology cycle around ∞_- or ∞_+. As a result, the mapping (2.3) has five vectors of periods in $\mathbb{C}^2$. Hence its inversion is not well defined. Namely, there are no meromorphic functions on $\mathbb{C}^2$ with five periods. In particular, $U(x, t)$ is not a meromorphic or single-valued complex function of (t, x).

In order to describe properties of $U(x, t)$, we fix time by putting $t = t_0$ ($\mathrm{d}t = 0$) and consider stationary solutions $U(x, t_0)$. Now introduce a new coordinate x' such that

$$\mathrm{d}x = \mu_1\mu_2\,\mathrm{d}x'. \tag{2.4}$$

Then equations (2.2) lead to the Abel–Jacobi mapping of $\Gamma^{(2)}$ to the Jacobian variety $\mathrm{Jac}(\Gamma)$ of Γ, which includes holomorphic differentials only:

$$\begin{aligned}
\int_{P_0}^{P_1} \frac{\mathrm{d}\mu}{2\sqrt{R_6(\mu)}} + \int_{P_0}^{P_2} \frac{\mathrm{d}\mu}{2\sqrt{R_6(\mu)}} &= u_1 \\
\int_{P_0}^{P_1} \frac{\mu\,\mathrm{d}\mu}{2\sqrt{R_6(\mu)}} + \int_{P_0}^{P_2} \frac{\mu\,\mathrm{d}\mu}{2\sqrt{R_6(\mu)}} &= u_2 \\
u_1 = x' + \text{const} \qquad u_2 &= \text{const}
\end{aligned} \tag{2.5}$$

where u_1 and u_2 are coordinates on the universal covering $\mathbb{C}^2$ of $\mathrm{Jac}(\Gamma)$.

Let $\bar{\omega}_1, \bar{\omega}_2$ be the dual basis of *normalized* holomorphic differentials on Γ with respect to the choice of cycles described above and z_1 and z_2 be the corresponding coordinates on the universal covering of $\mathrm{Jac}(\Gamma)$:

$$\begin{aligned}
\bar{\omega}_1 &= \frac{d_{11} + d_{12}\mu}{2\sqrt{R_6(\mu)}}\,\mathrm{d}\mu \qquad & \bar{\omega}_2 &= \frac{d_{21} + d_{22}\mu}{2\sqrt{R_6(\mu)}}\,\mathrm{d}\mu \\
z_1 &= d_{11}u_1 + d_{12}u_2 \qquad & z_2 &= d_{21}u_1 + d_{22}u_2.
\end{aligned} \tag{2.6}$$

Here the normalizing constants d are uniquely determined by the conditions $\oint_{A_i} \bar{\omega}_j = \delta_{ij}$.

Recall that the standard theta-function related to a Riemann surface of genus g and theta-functions with characteristics $\alpha = (\alpha_1, \ldots, \alpha_g)$, $\beta = (\beta_1, \ldots, \beta_g) \in \mathbb{R}^g$ have the form

$$\begin{gathered} \theta(z|\boldsymbol{B}) = \sum_{M\in\mathbb{Z}^g} \exp(\tfrac{1}{2}(\boldsymbol{B}M, M) + (M, z)) \\ (M, z) = \sum_{i=1}^{g} M_i z_i \qquad (\boldsymbol{B}M, M) = \sum_{i,j=1}^{g} \boldsymbol{B}_{ij} M_i M_j \\ \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\boldsymbol{B}) = \exp\{(\boldsymbol{B}\alpha, \alpha)/2 + (z + 2\pi \mathrm{i}\beta, \alpha)\}\theta(z + 2\pi\mathrm{i}\beta + \boldsymbol{B}\alpha|\boldsymbol{B}) \end{gathered} \tag{2.7}$$

where $\boldsymbol{B}$ is the $g \times g$ period matrix of Γ. In what follows we shall omit it. Now we choose the basepoint P_0 of the mapping (2.5) to be the last Weierstrass point $(a_5, 0)$ on Γ. Then, the trace formula for even-order hyperelliptic curves (see, e.g., Clebsch and Gordan 1866, Dubrovin 1981) yields the following formula:

$$\begin{gathered} U = \mu_1 + \mu_2 - \sum_{j=1}^{5} a_j = \text{const} - \partial_W \log \frac{\theta[\delta](z - q/2)}{\theta[\delta](z + q/2)} \\ z = (z_1, z_2) \qquad q = (q_1, q_2)^T \qquad q_i = \int_{\infty_-}^{\infty_+} \bar{\omega}_i \end{gathered} \tag{2.8}$$

where, in view of the normalizing change (2.6), one has that $z_1 = d_{11}x' + \text{const}$, $z_2 = d_{21}x' + \text{const}$. Also, ∂_W denotes a derivative along a tangent vector W of $\Gamma \subset \mathrm{Jac}(\Gamma)$ at ∞_+ which is $W = (0, 1)^T$ and $W = (d_{12}, d_{22})^T$ in coordinates (u_1, u_2) and (z_1, z_2) respectively. Finally, $(\delta = (\delta'', \delta')^T$; $\delta'', \delta' \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g)$ is a half-integer theta-characteristic which corresponds to a vector of Riemann constants (see Mumford 1983). In the case of a standard canonical basis of cycles chosen above, and a basepoint $P_0 = (a_5, 0)$ one has

$$\delta' = (1/2, \ldots, 1/2)^T \qquad \delta'' = (g/2, (g-1)/2, \ldots, 1, 1/2)^T \pmod 1. \tag{2.9}$$

Thus, in our case one has

$$\delta = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

The function $U(z_1, z_2)$ is meromorphic on $\mathrm{Jac}(\Gamma)$ and it has simple poles along two translates of the theta-divisor $\Theta = \{\theta(z) = 0\} \subset \mathrm{Jac}(\Gamma)$:

$$\Theta_- = \{\theta[\delta](z - q/2) = 0\} \qquad \Theta_+ = \{\theta[\delta](z + q/2) = 0\}$$

which are tangent to each other at the origin $\{z = 0\}$. Thus, $U(z_1(x'), z_2(x'))$ is a quasi-periodic function of a complex variable x'. Notice that a quasi-periodic genus 2 solution of the nonlinear mKdV equation has the same form.

We also notice that a point $E_0 = (\mu = 0, w = 0)$ is a Weierstrass (branch) point on Γ. Then, following Clebsch and Gordan (1866), we have the following expression for the symmetric polynomial:

$$\mu_1\mu_2 = \varrho \frac{\theta^2[\delta + \eta_0](z)}{\theta[\delta](z + q/2)\theta[\delta](z - q/2)} \qquad \varrho = \text{const} \tag{2.10}$$

where η_0 is a half-integer theta-characteristic corresponding to the branch point E_0:

$$\eta_0 = (\eta_0'', \eta_0')^T \qquad \int_{P_0}^{E_0} (\bar{\omega}_1, \bar{\omega}_2)^T = 2\pi\mathrm{i}\eta_0'' + \boldsymbol{B}\eta_0' \in \mathbb{C}^2. \tag{2.11}$$

Thus, the product $\mu_1\mu_2$ is a meromorphic function on $\mathrm{Jac}(\Gamma)$ having simple poles along Θ_-, Θ_+ and a *double* zero along another translate of the theta-divisor Θ, $\Theta_0 = \{\theta[\delta + \eta_0](z) = 0\}$, and passing through the origin and intersecting each of the translates Θ_-, Θ_+ at two points. The translate Θ_0 can also be interpreted as an image of the curve Γ under the action of the Abel–Jacobi mapping (2.5):

$$\Theta_0 = \left\{ \int_{P_0}^{P} (\bar\omega_1, \bar\omega_2)^T + \int_{P_0}^{E_0} (\bar\omega_1, \bar\omega_2)^T \,\middle|\, P \in \Gamma \right\}.$$

It follows from (2.4) and (2.10) that generically the derivative of the function $x(x')$ is equal to $\mu_1\mu_2$ and that it has a double zero each time the complex x'-flow intersects Θ_0, i.e. when $\theta[\delta+\eta_0](z)$ vanishes, except at the points where the flow is tangent to Θ_0, i.e. when $\theta[\delta+\eta_0](z)$ has a higher vanishing order in both x' and $\mu_1\mu_2$. Let z_0 denote coordinates of a point on Θ_0 and x_0', x_0 denote the corresponding values of x' and x respectively. Then a function $x(x') - x_0$ has a triple zero at x_0' and

$$x' - x_0' = \mathrm{O}((x - x_0)^{1/3}). \tag{2.12}$$

On the other hand, in view of the second equation of (2.3), the original variable x is a sum of Abelian integrals of the third kind. Introduce the normalized differentials of third kind $\Omega_{\infty_-\infty_+}$ on Γ having poles at ∞_-, ∞_+ with residues ± 1:

$$\Omega_{\infty_-\infty_+} = \frac{\mu^2\,\mathrm{d}\mu}{\sqrt{R_6(\mu)}} + h_1\bar\omega_1 + h_2\bar\omega_2 \tag{2.13}$$

where h_1, h_2 are normalizing constants specified by $\Omega_{\infty_-\infty_+}$ having zero A-periods on Γ. According to Clebsch and Gordan (1866), one has

$$\int_{P_0}^{P_1} \Omega_{\infty_-\infty_+} + \int_{P_0}^{P_2} \Omega_{\infty_-\infty_+} = \log \frac{\theta[\delta](z + q/2)}{\theta[\delta](z - q/2)} + \text{const}. \tag{2.14}$$

Then, in view of the second equation in (2.2) and (2.13), we obtain

$$\begin{aligned} x(x') &= \log \frac{\theta[\delta](z + q/2)}{\theta[\delta](z - q/2)} - h_1 z_1 - h_2 z_2 + \text{const} \\ z_1 &= d_{11}x' + \text{const} \qquad z_2 = d_{21}x' + \text{const}. \end{aligned} \tag{2.15}$$

As a result, we expressed a stationary quasi-periodic solution U and the argument x in terms of the auxiliary complex variable x'. An algebraic geometrical structure of the general solution $U(x, t)$ and the behaviour of real-valued solutions will be considered in the next section.

Stationary quasi-periodic solutions for the Dym equation. Now we pass to the Dym equation (1.2) and seek its solutions in the form (2.1). In this case the variables μ_1 and μ_2 again change according to equations of the form (2.2) with the only difference being that the order of the polynomial defining the corresponding hyperelliptic curve is odd:

$$\begin{aligned} &\frac{\mu_1\,\mathrm{d}\mu_1}{2\sqrt{R_5(\mu_1)}} + \frac{\mu_2\,\mathrm{d}\mu_2}{2\sqrt{R_5(\mu_2)}} = \mathrm{d}t \\ &\frac{\mu_1^2\,\mathrm{d}\mu_1}{2\sqrt{R_5(\mu_1)}} + \frac{\mu_2^2\,\mathrm{d}\mu_2}{2\sqrt{R_5(\mu_2)}} = \mathrm{d}x \\ &R_5(\mu) = -\kappa\mu(\mu - a_1)\cdots(\mu - a_4). \end{aligned} \tag{2.16}$$

Hence the corresponding hyperelliptic curve $\Gamma = \{w^2 = R_5(\mu)\}$ has just one infinite point ∞. As a consequence, the equations (2.16) contain one holomorphic differential and one differential of the *second* kind.

As before, we first consider stationary solutions by putting $t = t_0$ ($\mathrm{d}t = 0$) and assuming $\kappa = 1$. Notice that under these conditions, (2.16) has the same structure as quadratures for the Jacobi problem of geodesics on a triaxial ellipsoid Q, where μ_1 and μ_2 play the role of ellipsoidal coordinates on Q, parameters (a_1, a_2, a_3) are the squares of the semi-axes of Q, a_4 is a constant of motion and x is the length along a geodesic.

After using the change of parameter (2.4), we arrive at the Abel–Jacobi mapping

$$\begin{aligned} &\int_{P_0}^{P_1} \frac{\mathrm{d}\mu}{2\sqrt{R_5(\mu)}} + \int_{P_0}^{P_2} \frac{\mathrm{d}\mu}{2\sqrt{R_5(\mu)}} = u_1 \\ &\int_{P_0}^{P_1} \frac{\mu\,\mathrm{d}\mu}{2\sqrt{R_5(\mu)}} + \int_{P_0}^{P_2} \frac{\mu\,\mathrm{d}\mu}{2\sqrt{R_5(\mu)}} = u_2 \\ &u_1 = x' + \mathrm{const} \qquad u_2 = \mathrm{const}. \end{aligned} \tag{2.17}$$

This change of parameter was first used by Weierstrass (1878) in order to find the theta-functional solution for the geodesic problem (see also Cewen 1990). Notice that he was interested only in describing the geometrical trajectories (geodesics) and did not consider dynamics along those trajectories. (Namely, he did not consider returning to the initial parametrization in the final formulae.)

We introduce normalized holomorphic differentials $\bar{\omega}_1, \bar{\omega}_2$ on Γ and a normalized differential of the second kind having a double pole at ∞

$$\Omega_\infty^{(1)} = \frac{\mu_i^2\,\mathrm{d}\mu_i}{2\sqrt{R_5(\mu_i)}} + h_1'\bar{\omega}_1 + h_2'\bar{\omega}_2. \tag{2.18}$$

Now use (2.6) for determining coordinates z_1, z_2 on the universal covering of $\mathrm{Jac}(\Gamma)$. The constants h_1', h_2' are uniquely determined by requiring $\Omega_\infty^{(1)}$ to have zero A-periods on Γ. Now, instead of the expressions (2.8) and (2.10), one has (see, e.g., Dubrovin 1981, Dubrovin *et al* 1985)

$$\begin{aligned} &U(x') = \mu_1 + \mu_2 = \mathrm{const} - \partial_V^2\theta[\delta](z) \\ &z_1 = d_{11}x' + \mathrm{const} \qquad z_2 = d_{21}x' + \mathrm{const} \end{aligned} \tag{2.19}$$

where ∂_V is a derivative along the tangent vector V of $\Gamma \in \mathrm{Jac}(\Gamma)$ at ∞: $V = (d_{12}, d_{22})^{\mathrm{T}}$. Also, the following expression holds:

$$\mu_1\mu_2 = \kappa\,\frac{\theta^2[\delta + \eta_0](z)}{\theta^2[\delta](z)} \qquad \kappa = \mathrm{const} \tag{2.20}$$

where the characteristic η_0 is defined in (2.11). In addition, in contrast to (2.14), the sum of Abelian integrals of the second kind has the form

$$\int_{P_0}^{P_1} \Omega_\infty^{(1)} + \int_{P_0}^{P_2} \Omega_\infty^{(1)} = \mathrm{const} - \partial_V \log\theta[\delta](z). \tag{2.21}$$

Comparing this with (2.18) yields the following analogue of the relation (2.15) between the parameters x and x':

$$\begin{aligned} &x(x') = -\partial_V \log\theta[\delta](z) - h_1 z_1 - h_2 z_2 + \mathrm{const} \\ &z_1 = d_{11}x' + \mathrm{const} \qquad z_2 = d_{21}x' + \mathrm{const}. \end{aligned} \tag{2.22}$$

Thus, we have expressed the stationary solution U and the argument x in terms of the auxiliary complex variable x'. Real-valued solutions defined by the above expressions will be considered in section 4.

3. Time-dependent quasi-periodic solutions

The quasi-periodic solutions for the SW equation. In order to obtain general time-dependent solutions $U(x,t)$ of the SW equation given by (2.1), one has to invert the mapping (2.3). However, as already mentioned, the problem of inversion cannot be solved in terms of meromorphic functions.

To describe the structure of general solutions, let us first consider a divisor of three points $P_i = (\mu_i, w_i)$, $i = 1, 2, 3$ on $\Gamma \setminus \{\infty_-, \infty_+\}$ and the following extended equations:

$$\sum_{i=1}^{3} \frac{\mathrm{d}\mu_i}{2\sqrt{R_6(\mu_i)}} = \mathrm{d}y \qquad \sum_{i=1}^{3} \frac{\mu_i\,\mathrm{d}\mu_i}{2\sqrt{R_6(\mu_1)}} = \mathrm{d}t \qquad \sum_{i=1}^{3} \frac{\mu_i^2\,\mathrm{d}\mu_i}{2\sqrt{R_6(\mu_i)}} = \mathrm{d}x \tag{3.1}$$

which include an extra variable y, two holomorphic differentials and one differential of the third kind on Γ. The latter are linear combinations of the normalized differentials $\bar{\omega}_1, \bar{\omega}_2, \Omega_{\pm\infty}$ defined in (2.6) and (2.13). According to Clebsch and Gordan (1866), equations (3.1) describe a differential of a well defined mapping of the symmetric product $(\Gamma \setminus \{\infty_-, \infty_+\})^{(3)}$ to a *generalized Jacobian variety* $\mathrm{Jac}(\Gamma, \infty_\pm)$, a *noncompact* algebraic group represented in the form of a quotient of $\mathbb{C}^3$ by a lattice Λ generated by five vectors of periods of the differentials $\bar{\omega}_1, \bar{\omega}_2, \Omega_{\pm\infty}$ on Γ. Topologically, $\mathrm{Jac}(\Gamma, \infty_\pm)$ is a product of a two-dimensional variety $\mathrm{Jac}(\Gamma)$ and a cylinder $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. An analytical and algebraic–geometrical description of generalized Jacobians can be found in Clebsch and Gordan (1866), Belokolos *et al* (1994), Fedorov (1999) and Gavrilov (2000).

Let (z_1, z_2, Z) be coordinates on the universal covering of $\mathrm{Jac}(\Gamma, \infty_\pm)$ such that

$$\sum_{i=1}^{3} \int_{P_0}^{P_i} \bar{\omega}_1 = z_1 \qquad \sum_{i=1}^{3} \int_{P_0}^{P_i} \bar{\omega}_2 = z_2 \qquad \sum_{i=1}^{3} \int_{P_0}^{P_i} \Omega_{\pm\infty} = Z \tag{3.2}$$

where, as above, $P_0 = (a_5, 0)$. Then, (2.6) and (2.13) yield

$$\begin{aligned} &z_1 = d_{11}y + d_{12}t + \text{const} \qquad z_2 = d_{21}y + d_{22}t + \text{const} \\ &Z = x + h_1(d_{11}y + d_{12}t) + h_2(d_{21}y + d_{22}t) + \text{const}. \end{aligned} \tag{3.3}$$

The problem of inversion of Abel–Jacobi mappings which includes differentials of the third and second kinds is solved in terms of *generalized theta-functions* which are finite sums of products of customary theta-functions, rational functions and exponentials (see Ercolani 1989, Fedorov 1999, Gagnon *et al* 1985). To invert the mapping (3.2), we shall make use of the following theta-functions:

$$\begin{aligned} &\tilde{\theta}(z, Z) = \mathrm{e}^{Z/2}\theta(z + q/2) + \mathrm{e}^{-Z/2}\theta(z - q/2) \\ &\tilde{\theta}[\eta](z, Z) = \mathrm{e}^{Z/2}\theta[\eta](z + q/2) + \mathrm{e}^{-Z/2}\theta[\eta](z - q/2) \\ &z = (z_1, z_2) \qquad q = (q_1, q_2)^T \qquad q_1 = \int_{\infty_-}^{\infty_+} \bar{\omega}_1 \qquad q_2 = \int_{\infty_-}^{\infty_+} \bar{\omega}_2 \end{aligned} \tag{3.4}$$

where $\theta(z)$ and $\theta[\eta](z)$ are customary theta-functions associated with the curve Γ with half-integer theta-characteristics η. Like $\theta[\eta](z)$, generalized theta-functions have a quasi-periodic property: a shift of the argument (z, Z) by any period vector of the generalized Jacobian results in multiplication of $\tilde{\theta}[\eta](z, Z)$ by a constant factor.

Now consider the dissection $\tilde{\Gamma}$ of Γ along the canonical cycles A_1, A_2, B_1, B_2, which is a domain having the form of an octagon. In addition, we cut $\tilde{\Gamma}$ along the paths joining a point O on the boundary $\partial\tilde{\Gamma}$ of $\tilde{\Gamma}$ to the points ∞_-, ∞_+. On the resulting domain $\tilde{\Gamma}'$ we introduce a single-valued function $\tilde{F}(P) = \tilde{\theta}[\delta](\tilde{\mathcal{A}}(P) - (z, Z)^T)$, where

$$\tilde{\mathcal{A}}(P) = \left(\int_{P_0}^{P} \bar{\omega}_1, \int_{P_0}^{P} \bar{\omega}_2, \int_{P_0}^{P} \Omega_{\infty_\pm} \right)^T$$

and the characteristic δ is defined in (2.9). Then the following analogue of the Riemann theorem holds (see, e.g., Fedorov 1999, Gagnon *et al* 1985).

Theorem 3.1. *Let the coordinates z, Z be such that the function $\tilde{F}(P)$ does not vanish identically on $\tilde{\Gamma}'$. Then it has precisely three zeros P_1, P_2, P_3, which determine a unique solution to the problem of inverting the generalized mapping (3.2).*

Now let us consider the logarithmic differential $\mu(P)\,\mathrm{d}\log\tilde{F}(P)$. Theorem 3.1 results in the sum of residues of its poles in the domain $\tilde{\Gamma}'$ being equal to $\mu(P_1)+\mu(P_2)+\mu(P_3)$. After applying the residue theorem one obtains the following compact 'trace formula':

$$\mu_1+\mu_2+\mu_3=\mathrm{const}-\frac{\mathrm{e}^{Z}\theta[\delta](z+q)+\mathrm{e}^{-Z}\theta[\delta](z-q)}{\theta[\delta](z)} \tag{3.5}$$

with the characteristic δ specified in (2.9).

The principal difference between the extended mappings (3.1) or (3.2) and the system (2.3) is that the latter contains only two points on $\Gamma\setminus\{\infty_-,\infty_+\}$. On the other hand, (3.1) reduces to (2.2) by fixing $P_3\equiv P_0$ ($\mu_3\equiv a_5$, $\mathrm{d}\mu_3\equiv 0$). Under this condition, (3.2) describes the embedding of the symmetric product $(\Gamma\setminus\{\infty_-,\infty_+\})^{(2)}$ into $\mathrm{Jac}(\Gamma,\infty_\pm)$. Its image is a two-dimensional *nonlinear* analytic subvariety (stratum) W_2. Like the generalized Jacobian itself, it is a noncompact variety.

Remark 3.2. In the case of customary Jacobian varieties, corresponding nonlinear subvarieties and their stratification have been studied by Gunning (1972) and Vanhaecke (1995). Such varieties or their open subsets often appear as (coverings of) complex invariant manifolds of finite-dimensional integrable systems (see Vanhaecke 1995, Abenda and Fedorov 2000).

It follows from the above that on the stratum W_2 the variables z_1, z_2, Z play a role of excessive (abundant) coordinates. Hence they cannot be independent there. An analytic structure of W_2 is explicitly described by the following theorem (see, e.g., Fedorov 1999, Gagnon *et al* 1985).

Theorem 3.3. *The subvariety $W_2\subset\mathrm{Jac}(\Gamma,\infty_\pm)$ coincides with the zero locus of the generalized theta-function:*

$$W_2=\{\mathrm{e}^{Z/2}\theta[\delta](z+q/2)-\mathrm{e}^{-Z/2}\theta[\delta](z-q/2)=0\}. \tag{3.6}$$

On the other hand, in view of relations (3.3), the coordinates z, Z are linear functions of the variables x, t and y. Thus, equation (3.6) can be regarded as a constraint on them. It follows that after fixing $P_3=P_0$, y becomes a transcendental function of x, t.

Notice that the sum $\mu_1+\mu_2+a_5=\mu(P_1)+\mu(P_2)+\mu(P_0)$, considered as a function on $\mathrm{Jac}(\Gamma,\infty_\pm)$, coincides with the restriction to W_2 of the sum $\mu(P_1)+\mu(P_2)+\mu(P_3)$. Then, using expression (3.5), one concludes that the two-phase solution of the SW equation has the form

$$\begin{aligned}
&U(x,t)=\mathrm{const}-\frac{\mathrm{e}^{Z}\theta[\delta](z+q)+\mathrm{e}^{-Z}\theta[\delta](z-q)}{\theta[\delta](z)}\\
&z_1=d_{11}y+d_{12}t \qquad z_2=d_{21}y+d_{22}t\\
&Z=x+h_1(d_{11}y+d_{12}t)+h_2(d_{21}y+d_{22}t)
\end{aligned} \tag{3.7}$$

where an extra variable y depends on x and t according to (3.6). As a result, we arrive at the following algebraic–geometric description of motion:

The x-flow and t-flow defined by equations (2.2) evolve on the nonlinear variety $W_2\subset$ $\mathrm{Jac}(\Gamma,\infty_\pm)$ *in such a way that y is a nonlinear transcendental function of x and t respectively. This is why the flow is called a nonlinear flow.*

We emphasize that the solution $U(x,t)$ is meromorphic neither in x, nor in t.

Remark 3.4. Let us consider the x-flow by setting t =const. It turns out that, up to an additive constant, the extra variable y can now be identified with the auxiliary variable x' introduced in (2.4), where we considered stationary solutions. Indeed, in view of (3.3), in this case the condition in (3.6) becomes

$$Z = x + h_1 z_1 + h_2 z_2 + \text{const} = \log \frac{\theta[\delta](z - q/2)}{\theta[\delta](z + q/2)} + \text{const} \tag{3.8}$$

which is equivalent to the relation (2.15) between x and x'. In view of (3.8) and the addition theorem for theta-functions, the solution (3.7) reduces to the stationary solution (2.8).

In contrast to x, the parameter t enters both expressions for z and Z in (3.7). Therefore, in the case of the t-flow, t cannot be explicitly expressed in terms of y as is the case for the x-flow. This implies that solutions $U(x_0, t)$, $x_0 =$ const, must have different properties in comparison with (2.8).

The quasi-periodic solutions for the Dym equation. Now we proceed to the problem of inversion (2.16) associated with the Dym-type equation which is related to the odd-order hyperelliptic curve $\Gamma = \{w^2 = R_5(\mu)\}$. As in the case of finite-dimensional reduction of a class of quasi-periodic solutions of the SW equation, in order to describe the function $U(x, t) = \mu_1 + \mu_2$, we first consider an 'excessive' divisor of three points $P_i = (\mu_i, w_i)$, $i = 1, 2, 3$ on $\Gamma \setminus \{\infty\}$ and the extended system of equations in the form

$$\sum_{i=1}^{3} \frac{\mathrm{d}\mu_i}{2\sqrt{R_5(\mu_i)}} = \mathrm{d}y \qquad \sum_{i=1}^{3} \frac{\mu_i\,\mathrm{d}\mu_i}{2\sqrt{R_5(\mu_1)}} = \mathrm{d}t \qquad \sum_{i=1}^{3} \frac{\mu_i^2\,\mathrm{d}\mu_i}{2\sqrt{R_5(\mu_i)}} = \mathrm{d}x$$
$$R_5(\mu) = -\kappa\mu(\mu - a_1)\ldots(\mu - a_4) \tag{3.9}$$

which include two holomorphic differentials and one differential of the second kind having a double pole at $\infty \in \Gamma$. These differentials are linear combinations of the normalized differentials $\bar{\omega}_1, \bar{\omega}_2, \Omega_\infty^{(1)}$ defined in (2.6) and (2.18).

In contrast to (3.1), equations (3.9) describe a differential of a well defined mapping of the symmetric product $(\Gamma \setminus \{\infty\})^{(3)}$ to the generalized Jacobian variety $\mathrm{Jac}(\Gamma, \infty)$, the quotient of $\mathbb{C}^3$ by the lattice generated by *four* period vectors of the differentials $\bar{\omega}_1, \bar{\omega}_2, \Omega_\infty^{(1)}$ on Γ. Topologically this variety is a product of a two-dimensional variety $\mathrm{Jac}(\Gamma)$ and complex plane $\mathbb{C}$ (see Clebsch and Gordan 1866, Gavrilov 2000).

Let us introduce coordinates z_1, z_2, Z as follows:

$$\sum_{i=1}^{3} \int_{E_0}^{P_i} \bar{\omega}_1 = z_1 \qquad \sum_{i=1}^{3} \int_{E_0}^{P_i} \bar{\omega}_2 = z_2 \qquad \sum_{i=1}^{3} \int_{E_0}^{P_i} \Omega_\infty^{(1)} = Z \tag{3.10}$$

where the basepoint is chosen as $E_0 = (0, 0)$. We cannot choose the basepoint to be ∞ as in the previous section, since it is a pole of $\Omega_\infty^{(1)}$. Next, comparing (2.6) and (2.18) with (3.9) yields the following relations:

$$\begin{aligned} &z_1 = d_{11}y + d_{12}t + \text{const} \qquad z_2 = d_{21}y + d_{22}t + \text{const} \\ &Z = x + h_1'(d_{11}y + d_{12}t) + h_2'(d_{21}y + d_{22}t) + \text{const.} \end{aligned} \tag{3.11}$$

The mapping (3.10) is invertible in terms of meromorphic functions. The inversion problem is solved by means of the following rational degeneration of the customary theta-function:

$$\hat{\theta}(z, Z) = Z\theta[\delta](z) + \partial_V \theta[\delta](z) \tag{3.12}$$

where ∂_V is defined in (2.19) (compare with the generalized theta-functions (3.4)). The function (3.12) enjoys the quasi-periodic property.

Consider again the dissection $\tilde{\Gamma}$ of Γ and cut it along a path joining a point O on the boundary $\partial\Gamma$ to ∞. In the resulting domain we introduce a single-valued function

$$\hat{F}(P) = \left(Z - \int_{E_0}^{P} \Omega_\infty^{(1)}\right)\theta[\delta]\left(z - \int_{E_0}^{\infty} \bar{\omega} - \int_{E_0}^{P} \bar{\omega}\right) + \partial_V\theta[\delta]\left(z - \int_{E_0}^{\infty} \bar{\omega} - \int_{E_0}^{P} \bar{\omega}\right).$$

Using a modification of theorem 3.1 and calculating the logarithmic differential $\mu(P)\,\mathrm{d}\log\,\hat{F}(P)$, one obtains that

$$\begin{aligned}\mu_1 + \mu_2 + \mu_3 &= \text{const} - (Z + \partial_V \log\theta[\delta + \eta_0](z))^2 - \partial_V^2 \log\theta[\delta + \eta_0](z)\\ &= \text{const} - Z^2 - \frac{2Z\partial_V\theta[\delta + \eta_0](z) - \partial_V^2\theta[\delta + \eta_0](z)}{\theta[\delta + \eta_0](z)}\end{aligned} \qquad (3.13)$$

where $\eta_0 = (\eta_0'', \eta_0')^T \in \frac{1}{2}\mathbb{Z}^2/\mathbb{Z}^2$ is chosen such that $2\pi\mathrm{i}\eta_0'' + \boldsymbol{B}\eta_0' = \int_{E_0}^{\infty}(\bar{\omega}_1, \bar{\omega}_2)^T$. Now, similarly to the case of the SW equation, we fix $P_3 \equiv E_0$ ($\mu_3 = 0$, $\mathrm{d}\mu_3 \equiv 0$) in the mapping (3.10). Its image becomes a two-dimensional nonlinear noncompact analytical subvariety $\hat{W}_2 \subset \mathrm{Jac}(\Gamma, \infty)$. Comparing the third sum in (3.10) and expression (2.21) yields

$$\hat{W}_2 = \{Z + \text{const} + \partial_V \log\theta[\delta + \eta_0](z) = 0\}. \qquad (3.14)$$

Finally, taking into account the trace formula (3.13) we conclude that the solution of the Dym equation has the form

$$\begin{aligned}&U(x, t) = \mu_1 + \mu_2 = \text{const} - \partial_V^2 \log\theta[\delta + \eta_0](z)\\ &z_1 = d_{11}y + d_{12}t + \text{const} \qquad z_2 = d_{21}y + d_{22}t + \text{const}\end{aligned} \qquad (3.15)$$

where an extra variable y, according to the constraint (3.14) and an expression for Z in (3.11), depends on (x, t) in a transcendental way. The solution $U(x, t)$ is not meromorphic with respect to its arguments.

Remark 3.5. As in the case of the SW equation, the stationary solutions for the Dym-type equation given in the previous section can be obtained from (3.15) by setting t =const. Then y can be identified with an auxiliary variable x' defined in (2.4) and the condition in (3.14) becomes equivalent to the relation (2.22) between x and x'. As a result, (3.15) gives precisely the stationary solution (2.19).

The g-phase solutions as well as their different singular limiting cases are described in detail by Alber and Fedorov (2000).

Real bounded stationary solutions. The branching of *complex* stationary solutions of the SW and Dym-type equations may lead to existence of cusps in their *real-valued* solutions viewed as functions of $x \in \mathbb{R}$.

Let σ be an antiholomorphic involution on the genus n curve $\Gamma = \{w^2 = \mu R(\mu)\}$. The part of Γ which is invariant with respect to σ is called a real part $\Gamma_{\mathbb{R}}$. On the plane $\mathbb{R}^2 = (\operatorname{Re}\mu, \operatorname{Re} w)$ it is either an empty set or a union of ovals. By using an Abel–Jacobi mapping the involution σ lifts to Jac(Γ). By $\mathrm{Jac}_{\mathbb{R}}(\Gamma)$ we denote the real part of Jac(Γ) that is invariant under σ. The elementary symmetric functions of the variables $\mu_1, \ldots, \mu_n$ take real values on $\mathrm{Jac}_{\mathbb{R}}(\Gamma)$ and only there. Notice that the variables themselves are not necessarily real: some of them (or all) may be complex conjugate.

Suppose all the roots of the polynomial $R(\mu)$ are real and positive, i.e. $\Gamma_{\mathbb{R}}$ consists of $s = n$ or $n + 1$ ovals. According to a theorem by Comessatti (1924), there are 2^{n-1} connected components of $\mathrm{Jac}_{\mathbb{R}}(\Gamma)$ on which the elementary symmetric functions of the μ-variables are finite. The components are divided into two classes characterized by different behaviour of real solutions $U(x, t_0)$.

Case 1. One of the μ-variables evolves in the interval $[0, a_1]$, whereas other variables evolve in intervals $[a_j, a_{j+1}]$ or come in complex conjugate pairs. The sum $\mu_1 + \cdots + \mu_n$ is a real-valued quasi-periodic function of x' having no poles, whereas the product $\mu_1 \ldots \mu_n$ must have zeros in x'. These are generically double zeros, and the function $x(x')$ behaves in accordance with

$$x' - x'_0 = \mathrm{O}((x - x_0)^{1/3}) \tag{3.16}$$

which implies that the graph of $U(x, t_0)$ has a cusp. Due to quasi-periodicity of evolution of the μ-variable in the interval $[0, a_1]$, there is an infinite number of such cusps.

Case 2. None of the μ-variables evolves in the interval $[0, a_1]$. Then the product $\mu_1 \ldots \mu_n$ never vanishes. In view of (2.4), $x(x')$ and $x'(x)$ are strictly monotonic real smooth functions. Therefore composition $U(x, t_0) = U(x'(x))$ is a quasi-periodic smooth function. As a result, we proved the following theorem.

Theorem 3.6. *A real-valued quasi-periodic stationary solution $U(x, t_0)$ to the HD and SW equations is either a smooth quasi-periodic function or a function containing an infinite quasi-periodic sequence of cusps.*

4. Singular limits of the quasi-periodic solutions

Here we study certain degenerate cases of the quasi-periodic solutions of which the behaviour is similar to that of solitons. They are obtained after pinching to points *all* A-cycles on the associated hyperelliptic Riemann surface $\Gamma = \{w^2 = \mu R(\mu)\}$. The solutions are then expressed in terms of purely exponential tau-functions and, in the real bounded case, they describe an interaction of several smooth solitons or cuspons.

Shallow-water equation. Let us consider the following limiting form of a polynomial:

$$R(\mu) = (\mu - a)(\mu - b_1)^2 \ldots (\mu - b_n)^2$$

where $a, b_1, \ldots, b_n$ are arbitrary nonzero constants. Then the system of equations

$$\sum_{i=1}^{n} \frac{\mu_i^k \, \mathrm{d}\mu_i}{2(\mu_i - b_1) \ldots (\mu_i - b_n)\sqrt{\mu_i(\mu_i - a)}} = \begin{cases} 0 & k = 1, \ldots, n-2 \\ \mathrm{d}t & k = n-1 \\ \mathrm{d}x & k = n \end{cases} \tag{4.1}$$

describes a class of soliton-type solutions of the SW equation. This system involves only differentials of the third kind on the genus 0 surface $\mathbb{P} = \{w^2 = \mu(\mu - a)\}$ which have pairs of simple poles at the points Q_i^-, Q_i^+ with the coordinates $\mu_i = b_i$ and at the infinite points ∞_-, ∞_+.

Let us introduce corresponding normalized differentials of the third kind

$$\Omega_i = \frac{\sqrt{b_i(b_i - a)}\, \mathrm{d}\mu}{(\mu - b_i)w} \qquad i = 1, \ldots, n \qquad \Omega_{\pm\infty} = \frac{\mathrm{d}\mu}{w} \tag{4.2}$$

each of them having one pair of poles at the points described above with residue ± 1. Notice that the left-hand sides of (4.1) are linear combinations of the $n + 1$ differentials (4.2) with constant coefficients depending on a, b_i.

We first extend the system (4.1) by introducing an extra variable y such that

$$\sum_{i=1}^{n} \frac{\mathrm{d}\mu_i}{2(\mu_i - b_1) \ldots (\mu_i - b_n)\sqrt{\mu_i(\mu_i - a)}} = \mathrm{d}y. \tag{4.3}$$

Next, instead of n points, we consider the divisor of $n+1$ points on $\mathbb{P}\setminus\{Q_i^\pm, \infty_\pm\}$ and represent the extension of the system (4.1) and (4.3) in the integral form which involves the following normalized differentials:

$$\sum_{k=1}^{n+1}\int_{P_0}^{P_k}\Omega_i = z_i \qquad k=1,\ldots,n \qquad \sum_{i=1}^{n+1}\int_{P_0}^{P_i}\Omega_{\pm\infty} = z_{n+1} \tag{4.4}$$

where we set $P_0 = (a, 0) \in \mathbb{P}$. After comparing (4.4), (4.2) with (4.1), (4.3) we find that

$$\begin{aligned} z_i &= 2\sqrt{b_i(b_i-a)}\left(t - (-1)^n \frac{b_1\ldots b_n}{b_i} y\right) + \text{const} \qquad i=1,\ldots,n \\ z_{n+1} &= 2x - 2(b_1+\cdots+b_n)t + 2\,b_1\ldots b_n\, y + \text{const}. \end{aligned} \tag{4.5}$$

The system (4.4) represents a well defined mapping of a symmetric product $(\mathbb{P}\setminus\{Q_1^\pm,\ldots,Q_n^\pm,\infty_\pm\})^{(n+1)}$ to the generalized Jacobian variety denoted as $\mathrm{Jac}(\mathbb{P}, Q_i^\pm, \infty_\pm)$, which is now isomorphic to $(\mathbb{C}^*)^{n+1}$ (see, e.g., Mumford 1983).

To simplify calculations further, we apply a projective transformation $\mu = a\lambda/(\lambda-1)$ which maps branch points $\mu = 0$ and $\mu = a$ of $\mathbb{P}$ to zero and infinity respectively. As a result, the differentials (4.2) take the following form:

$$\begin{aligned} \Omega_i &= \frac{\alpha_i\,\mathrm{d}\lambda}{(\lambda-\alpha_i^2)\sqrt{\lambda}} \qquad \alpha_i = \sqrt{b_i/(b_i-a)} \qquad i=1,\ldots,n \\ \Omega_{\pm\infty} &= \Omega_{n+1} = \frac{\alpha_{n+1}\,\mathrm{d}\lambda}{(\lambda-\alpha_{n+1}^2)\sqrt{\lambda}} \qquad \alpha_{n+1} = 1. \end{aligned} \tag{4.6}$$

After integrating equations (4.4) and setting $\xi_i = \sqrt{\lambda_i}$, one obtains that

$$\frac{(\xi_1-\alpha_k)\ldots(\xi_{n+1}-\alpha_k)}{(\xi_1+\alpha_k)\ldots(\xi_{n+1}+\alpha_k)} = \mathrm{e}^{z_k} \qquad k=1,\ldots,n+1. \tag{4.7}$$

The problem of inversion of (4.7) is solvable by using a *standard* $(n+1)$-dimensional tau-function (see, e.g., Mumford 1983), which is a degenerate form of Riemann theta-function, or by using generalized theta-functions which we represent in the following symmetric form:

$$\begin{aligned} &\tau(z_1,\ldots,z_g|\alpha_1,\ldots,\alpha_g) = \sum_{\varepsilon_k=\pm1}\exp\{\tfrac{1}{2}(\varepsilon,z) + \tfrac{1}{2}(\varepsilon^T S\varepsilon)\} \\ &\varepsilon = (\varepsilon_1,\ldots,\varepsilon_g)^T \qquad S_{ij} = \frac{1}{4}\log\left(\frac{\alpha_i-\alpha_j}{\alpha_i+\alpha_j}\right)^2 \\ &S_{ii} = 0 \qquad i,j=1,\ldots,g \qquad g \geqslant 2. \end{aligned} \tag{4.8}$$

Notice that in our case $g = n+1$. The parameter $4S_{ij} = 4S_{ji}$ is an integral of the differential Ω_j taken along the path joining the poles of Ω_i on $\mathbb{P}$. In particular, the following analogue for tau-functions of the Matveev–Its formula holds (see, e.g., McKean 1979):

$$\lambda_1+\cdots+\lambda_g = \partial_V^2\log\tau(z,\alpha) + \alpha_1^2+\cdots+\alpha_g^2 \qquad \partial_V = 2\sum_{k=1}^{g}\alpha_k\frac{\partial}{\partial z_k}. \tag{4.9}$$

Also, as noticed by Hirota, the expression

$$u(x,t) = \partial_V^2\log\tau(z,\alpha) \qquad z_k = z_{k0} + \alpha_k x + 4\alpha_k^3 t \qquad z_{k0} = \text{const}$$

involving tau-function (4.8) provides a g-soliton solution to the KdV equation and describes interaction of g solitary waves on the line x.

Notice that the system (4.7) represents a system of linear equations with respect to the elementary symmetric functions of $\xi_1,\ldots,\xi_{n+1}$. Solving it, we arrive at the following lemma.

Lemma 4.1. *The following expressions in terms of tau-functions hold:*

$$
\begin{aligned}
\mu_1 + \cdots + \mu_{n+1} &\equiv a \sum_{i=1}^{n+1} \frac{\xi_i^2}{\xi_i^2 - 1} \\
&= a \frac{\mathrm{e}^{z_{n+1}} \tau_n(z + 4\bar{S}_n) + \mathrm{e}^{-z_{n+1}} \tau_n(z - 4\bar{S}_n)}{4\tau_n(z)} + \text{const} \qquad (4.10)\\
\bar{S}_n &= (S_{1,n+1}, \ldots, S_{n,n+1})^T \in \mathbb{C}^n \\
\sqrt{\mu_1, \ldots, \mu_{n+1}} &\equiv \xi_1 \ldots \xi_{n+1} = \frac{\tau(z_1 + \pi\mathrm{i}, \ldots, z_{n+1} + \pi\mathrm{i})}{\tau(z_1, \ldots, z_{n+1})} \qquad (4.11)
\end{aligned}
$$

where $\tau_n(z) = \tau(z_1, \ldots, z_n)$ is an n-dimensional tau-function.

Notice that the mapping (4.4) is reduced to the mapping defined by the union of (4.1) and (4.3), if we put, for example, $P_{n+1} \equiv (0, 0)$, which corresponds to $\xi_{n+1} \equiv 0$ in (4.7). Under this condition, the variables z_i become dependent. After setting $\xi_1 \ldots \xi_{n+1} = 0$ in (4.11), we obtain the following constraint:

$$
\tau(z_1 + \pi\mathrm{i}, \ldots, z_{n+1} + \pi\mathrm{i}) = 0. \qquad (4.12)
$$

This defines a *nonlinear* n-dimensional subvariety of the generalized Jacobian $\mathrm{Jac}(\mathbb{P}, Q_i^{\pm}, \infty_{\pm})$. As a result, by using the trace formula (4.10) and the linear dependence of z_i on (x, t, y) in (4.5), we arrive at the following theorem.

Theorem 4.2. *The soliton-like solution of the SW equation has the form*

$$
\begin{aligned}
U(x, t) &= a \frac{\mathrm{e}^{z_{n+1}} \tau_n(z + 4\bar{S}_n) + \mathrm{e}^{-z_{n+1}} \tau_n(z - 4\bar{S}_n)}{4\tau_n(z)} + \text{const} \\
\bar{S}_n &= (S_{1,n+1}, \ldots, S_{n,n+1})^T \\
z_i &= 2\sqrt{b_i(b_i - a)} \left(t - (-1)^n \frac{b_1 \ldots b_n}{b_i} y \right) + \text{const} \qquad i = 1, \ldots, n \\
z_{n+1} &= 2x - 2(b_1 + \cdots + b_n)t + 2 b_1 \ldots b_n y + \text{const}
\end{aligned}
\qquad (4.13)
$$

where the extra variable y depends on x, t according to (4.12).

Now let us assume without loss of generality that $a, b_1, \ldots, b_n \in \mathbb{R}$ and $b_1 < \cdots < b_n$. Let s be the number of constants b_i that are separated from zero by a. Then the properties of real-valued bounded solutions $U(x, t)$ are described by the following theorem.

Theorem 4.3. *For real x, t and the additive constants in (4.5), the solution (4.13) has no poles and is real as well. It describes an interaction of $\tilde{s}$ smooth solitary waves with $n - \tilde{s}$ solitary cusps which results in phase shifts after each interaction in a way similar to the n-soliton interaction for the KdV equation. Parameter $\tilde{s}$ depends on the values of the additive constants in (4.5):*

$$
\tilde{s} = s, s + 2, \ldots, s + 2[(n - s)/2].
$$

For details see Alber and Fedorov (2000).

Dym equation. We now concentrate on soliton-like solutions of the Dym-type equation associated with the following limiting form of an odd-order hyperelliptic curve Γ:

$$
R(\mu) = (\mu - b_1)^2 \ldots (\mu - b_n)^2
$$

$b_1, \ldots, b_n$ again being arbitrary nonzero constants. As we shall see below, corresponding expressions in terms of tau-functions have a structure different from the SW case. The differential equations which describe soliton-like solutions of the Dym-type equation have the form

$$\sum_{i=1}^{n} \frac{\mu_i^k \,\mathrm{d}\mu_i}{2(\mu_i - b_1)\dots(\mu_i - b_n)\sqrt{\mu_i}} = \begin{cases} 0 & k = 1, \dots, n-2 \\ \mathrm{d}t & k = n-1 \\ \mathrm{d}x & k = n. \end{cases} \tag{4.14}$$

This system includes meromorphic differentials defined on the genus 0 surface $\mathbb{P} = \{\xi^2 = \mu\}$ having pairs of simple poles (Q_i^-, Q_i^+), $i = 1, \dots, n$ with the coordinates $\mu = b_i$ respectively, and a double pole at $\infty \in \mathbb{P}$.

Let us introduce corresponding normalized differentials of the third and second kinds

$$\Omega_i = \frac{\beta_i \,\mathrm{d}\mu}{(\mu - b_i)\xi} \qquad \beta_i = \sqrt{b_i} \qquad i = 1, \dots, n \qquad \Omega_\infty^{(1)} = -\frac{\mathrm{d}\mu}{2\xi}. \tag{4.15}$$

Now extend the system of equations (4.14) by introducing an extra variable y as follows:

$$\sum_{i=1}^{n} \frac{\mathrm{d}\mu_i}{2(\mu_i - b_1)\dots(\mu_i - b_n)\sqrt{\mu_i}} = \mathrm{d}y. \tag{4.16}$$

As in the case of the SW equation, we first consider $n + 1$ points $P_i = (\mu_i, \xi_i)$ on

$$\mathbb{P}' = \mathbb{P} \setminus \{Q_1^-, Q_1^+, \dots, Q_n^-, Q_n^+, \infty\}$$

and the following extended sums of integrals:

$$\sum_{i=1}^{n+1} \int_{P_0}^{P_i} \Omega_i = z_i \qquad i = 1, \dots, n \qquad \sum_{i=1}^{n+1} \int_{P_0}^{P_i} \Omega_\infty^{(1)} = z_{n+1} \tag{4.17}$$

with the basepoint $P_0 = (0, 0)$. This describes a well defined invertible mapping of the symmetric product $(\mathbb{P}')^{(n+1)}$ to the generalized Jacobian $\mathrm{Jac}(\mathbb{P}, Q_i^\pm, \infty)$ which is isomorphic to the direct product of $\mathbb{C}$ and n copies of $\mathbb{C}^*$ and which is topologically different from that the generalized Jacobian arising in the case of the SW equation.

Comparing (4.14), (4.16) with (4.17), one again arrives at the relations (4.5) which describe linear dependence of z_i on x, t, and y.

Now integrating (4.17), we obtain the following relations:

$$\begin{aligned} &\frac{(\xi_1 - \beta_k)\dots(\xi_{n+1} - \beta_k)}{(\xi_1 + \beta_k)\dots(\xi_{n+1} + \beta_k)} = \mathrm{e}^{z_k} \qquad k = 1, \dots, n \\ &\xi_1 + \dots + \xi_{n+1} = -z_{n+1} \end{aligned} \tag{4.18}$$

which can be regarded as a system of linear algebraic equations with respect to the elementary symmetric functions of ξ_i. After solving this system we obtain the expression

$$\sum_{i=1}^{n+1} \mu_i = \sum_{k=1}^{n} \beta_k^2 - (z_{n+1} + \partial_W \tau_n(z))^2 - \partial_W^2 \tau_n(z) \tag{4.19}$$

$$\xi_1 \dots \xi_n = \beta_1 \dots \beta_n \,[z_{n+1} + \partial_W \tau_n(z)] \tag{4.20}$$

where $\tau_n(z)$ is an n-dimensional tau-function defined in (4.8) with $g = n$, α_k is replaced by β_k and $\partial_W = \sum_{k=1}^{n} \beta_k \,\partial/\partial z_k$.

Now we fix P_{n+1} in the mapping (4.17) by setting $\xi_{n+1} = 0$. Then, according to (4.20), we get a constraint on the variables $z_1, \dots, z_{n+1}$ in the form

$$z_{n+1} + \partial_W \tau_n(z) = 0. \tag{4.21}$$

This equation defines a nonlinear n-dimensional subvariety of the generalized Jacobian $\mathrm{Jac}(\mathbb{P}, Q_i^\pm, \infty)$.

As a result, comparing the expression (4.19) with the trace formula (1.3) and using the linear relations (4.5), we arrive at the following theorem.

Theorem 4.4. *Soliton-like solutions of the Dym-type equation are given by*

$$
\begin{aligned}
U(x,t) &= \sum_{k=1}^{n} \beta_k^2 - (z_{n+1} + \partial_W \tau_n(z))^2 - \partial_W^2 \tau_n(z) \\
z_i &= 2\sqrt{b_i(b_i - a)} \left(t - (-1)^n \frac{b_1 \dots b_n}{b_i} y \right) + \text{const} \qquad i = 1, \dots, n \\
z_{n+1} &= 2x - 2(b_1 + \dots + b_n)t + 2\, b_1 \dots b_n\, y + \text{const}
\end{aligned}
\tag{4.22}
$$

where an extra variable y depends on x, t according to the constraint equation (4.21).

Acknowledgments

The research of MSA was partially supported by NSF grant DMS 9626672. The research of YNF was supported by INTAS grant 9710771 and, in part, by the Center for Applied Mathematics, University of Notre Dame.

References

Abenda S and Fedorov Yu 2000 On the weak Kowalewski–Painlevé property for hyperelliptically separable systems *Acta Appl. Math.* **60** 137–78

Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)

Ablowitz M J and Ma Y-C 1981 The periodic cubic Schrödinger equation *Stud. Appl. Math.* **65** 113–58

Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)

Alber M S and Alber S J 1985 Hamiltonian formalism for finite-zone solutions of integrable equations *C. R. Acad. Sci., Paris* **301** 777–81

Alber M S, Camassa R, Fedorov Yu, Holm D D and Marsden J E 1999 On billiard solutions of nonlinear PDE's *Phys. Lett.* A **264** 171–8

——2000a The geometry of new classes of weak billiard solutions of nonlinear PDEs *Preprint*

Alber M S, Camassa R and Gekhtman M 2000b On billiard weak solutions of nonlinear PDE's and Toda flows *CRM Proc. and Lecture Notes* vol 25 pp 1–11

Alber M S, Camassa R, Holm D D and Marsden J E 1994 The geometry of peaked solitons and billiard solutions of a class of integrable PDE's *Lett. Math. Phys.* **32** 137–51

——1995 On the link between umbilic geodesics and soliton solutions of nonlinear PDE's *Proc. R. Soc.* **450** 677–92

Alber M S and Fedorov Yu 2000 Algebraic geometrical solutions for certain evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians submitted

Alber M S, Luther G G and Marsden J E 1997 Energy dependent Schrödinger operators and complex Hamiltonian systems on Riemann surfaces *Nonlinearity* **10** 223–41

Alber M S and Miller C 2001 On peakon solutions of the shallow water equation *Appl. Math. Lett.* **14** 93–8

Alber S J 1979 Investigation of equations of Korteweg–de Vries type by the method of recurrence relations *J. Lond. Math. Soc.* **19** 467–80 (in Russian)

Belokolos E D, Bobenko A I, Enol'sii V Z, Its A R and Matveev V B 1994 *Algebro-Geometric Approach to Nonlinear Integrable Equations (Springer-Verlag Series in Nonlinear Dynamics)* (Berlin: Springer)

Beals R, Sattinger D H and Szmigielski J 1999 Multi-peakons and a theorem of Stietjes *Inverse Problems* **15** L1–4

——2000 Multipeakons and the classical moment *Adv. Math.* at press

Calogero F 1995 An integrable Hamiltonian system *Phys. Lett.* A **201** 306–10

Calogero F and Francoise J-P 1996 A completely integrable Hamiltonian system *J. Math. Phys.* **37** 2863–71

Camassa R and Holm D D 1993 An integrable shallow water equation with peaked solitons *Phys. Rev. Lett.* **71** 1661–4

Camassa R, Holm D D and Hyman J M 1994 A new integrable shallow water equation *Adv. Appl. Mech.* **31** 1–33

Cewen C 1990 Stationary Harry-Dym's equation and its relation with geodesics on ellipsoid *Acta Math. Sin.* **6** 35–41

Clebsch A and Gordan P 1866 *Theorie der Abelschen Funktionen* (Leipzig: Teubner)

Comessatti A 1924 Sulle varietá abeliane reale I *Ann. Math. Pure Appl.* **2** 67–106

Dmitrieva L A 1993a Finite-gap solutions of the Harry Dym equation *Phys. Lett.* A **182** 65–70

——1993b The higher-times approach to multisoliton solutions of the Harry Dym equation *J. Phys. A: Math. Gen.* **26** 6005–20

Dubrovin B 1981 Theta functions and nonlinear equations *Russ. Math. Surv.* **36** 11–92

Dubrovin B A, Novikov S P and Krichever I M 1985 *Integrable Systems. I. Itogi Nauki i Tekhniki. Sovr. Probl. Mat. Fund. Naprav. 4, VINITI* (Moscow) (Engl. Transl. 1989 *Encyclopaedia of Mathematical Sciences* vol 4 (Berlin: Springer))

Ercolani N 1989 Generalized theta functions and homoclinic varieties *Proc. Symp. Pure Appl. Math.* **49** 87–100

Fedorov Yu 1999 Classical integrable systems and billiards related to generalized Jacobians *Acta Appl. Math.* **55** 151–201

Gagnon L, Harnad J, Hurtubise J and Winternitz P 1985 Abelian integrals and the reduction method for an integrable Hamiltonian system *J. Math. Phys.* **26** 1605–12

Gavrilov L 2000 Generalized Jacobians of spectral curves and completely integrable systems *Math. Z.* at press

Gunning R 1972 *Lectures on Riemann Surfaces. Jacobi Varieties* (Princeton, NJ: Princeton University Press)

Hunter J K and Zheng Y X 1994 On a completely integrable nonlinear hyperbolic variational equation *Physica* D **79** 361–86

Its A R and Kotlyarov V P 1976 Explicit formulas for solutions of a nonlinear Schrödinger equation *Dokl. Akad. Nauk* A *(Ukrain)* **11** 965–8 (in Russian)

Lax P 1975 Periodic solutions of the KdV equations *Commun. Pure Appl. Math.* **28** 141–88

Li Y A and Olver P J 1998 Convergence of solitary-wave solutions in a perturbed bi-Hamiltonian dynamical system *Discrete Continuous Dyn. Syst.* **4** 159–91

Markushevich A I 1977 *Theory of Functions of a Complex Variable* (New York: Chelsea)

McKean H P 1979 Integrable systems and algebraic curves *Global Analysis (Springer Lecture Notes in Mathematics vol 775)* pp 83–200

McKean H P and Constantin A 1999 A shallow water equation on the circle *Commun. Pure Appl. Math.* **52** 949–82

Mumford D 1983 *Tata Lectures on Theta II* (Basle: Birkhauser)

Newell A C 1985 *Solitons in Mathematical Physics* (Philadelphia, PA: SIAM)

Novikov D P 1999 Algebraic geometric solutions of the Harry Dym equations *Siber. Math. J.* **40** 136–40

Novikov S P 1974 A periodic problem for the Korteweg–de Vries equation *Funct. Anal. Appl.* **8** 54–66

Novikov S P, Manakov S V, Pitaevskiĭ L P and Zakharov V E 1984 *Theory of Solitons. The Inverse Scattering Method (Contemporary Soviet Mathematics. Consultants Bureau)* (New York: Plenum) (translated from Russian)

Previato E 1985 Hyperelliptic quasi-periodic and soliton solutions of the nonlinear Schrödinger equation *Duke Math. J.* **52** 329–77

Ragnisco O and Bruschi M 1996 Peakons, r-matrix and Toda lattice *Physica* A **228** 150–9

Suris Y B 1996 A discrete time peakons lattice *Phys. Lett.* A **217** 321–9

Vanhaecke P 1995 Integrable systems and symmetric products of algebraic curves *Math. Z.* **40** 143–72

Weierstrass K 1878 Über die geodätischen Linien auf dem dreiachsigen ellipsoid *Math. Werke* I 257–66